Energy consumption in cellular network: ON-OFF model and impact of mobility

Thanh Tung Vu
Telecom Paristech, LTCI
Paris, France

Laurent Decreusefond
Telecom Paristech, LTCI
Paris, France

Philippe Martins
Telecom Paristech, LTCI
Paris, France

***Abstract*—In this paper we present a new analysis of energy consumption in cellular network and we focus on the distribution of energy consumed by a base station. We first define the energy consumption model, in which the consumed energy is divided into two parts: the additive part and the broadcast part. The additive part is served to communicate with different users while the broadcast part is served to transmit the same information to all users in the cell. We then model user's activity as ON-OFF process in time. We also model user's mobility as a random process and we define the high mobility regime in which users move very fast. We are able to provide analytical expressions for statistics of consumed energy and bounds on the distribution of energy in motionless case. We then consider the impact of mobility and we mobility reduces moments and central moments of the additive part. We also show that high mobility regime, the variance of energy tends to zero. This is a strong result as it holds true for almost any mobility model. We are able to characterize the convergence decay rate of the variance in function of user's speed. We propose two applications of the model. The first one is to dimension cell radius in the economical point of view and the second one is to dimension cell battery in the sites that do not have access to electricity.**

I. Introduction

A. Motivation

Cellular communications have had a phenomenal progression due to recent technological advances in both cellular networks and cellular telephone manufacturing. They will experience even more growth in the next decade. Data traffic is becoming more and more dominant in mobile networks. For the first time in history, the volume of worldwide mobile data traffic exceeded that of voice traffic in December 2009 ([1]).

Such demands, on both number of subscribers and traffic capacity, are driving forces for the telecommunication industries to develop new technologies, and for operators to invest more and more in infrastructures. According to several studies, 0.5% of world-wide electrical energy is responsible to cellular network and 80% of this is consumed at base station sites. Up to 90% of cellular energy consumption is operator's operational expenditures (OPEX) ([2],[3],[4]). A BS connected to electrical grid may cost approximately 3000$ per year to operate, while an off-grid BS may cost ten times more. Considering the rising energy consumption of mobile networks, it becomes clear that energy cost is critical for operators' OPEX. The expense for energy to run cellular networks is expected to triple over the next seven years.

Consequently, one of the main objectives for new generation of cellular network is to reduce energy consumption and carbon emission and to improve energy utilization efficiency. That leads to the concept of "green cellular network". The interest of green cellular network is not limited to ecological reasons but also includes on economical benefits. We refer to [3], [5] and reference therein for green techniques for cellular network.

B. Contribution summary

In this paper, we investigate different aspects, which surprisingly have not been investigated in the literature. We are interested in the distribution of energy consumed by a base station during its operating period. As mobility is the main issue in cellular network, we also take it into account. More precisely, we model users' initial position as a spatial Poisson point process, users' activity as ON-OFF random process and users' motion as random processes. We define the energy consumption model in which the consumed energy consists of the additive part to communicate different information to different users and the broadcast part to transmit the same information to all users in the cell. We are able to provide analytical expressions for the moments of the additive part. We provide bounds on the distribution of the consumed energy and we show that mobility reduces central moments and moments of the additive part. We also show that in high mobility regime, the variance of consumed energy reduces to zero and we characterize the convergence rate in the function of users' speed. We apply the achieved results to solve two dimensioning problems. The first one is to dimension optimal cell radius and the second one is to dimension base station's battery.

The rest of the present paper is organized in the following way. In the section II we describes the model, including the ON-OFF model and the mobility model. In the section III we provide analytical results for the case of motionless users. In the section IV we analyze the case where users move. In the section V we refine some results in the section IV to the case where users' speed is constant. In the section VI we consider two applications of the model: dimensioning cell radius and dimensioning base station's battery. Here is the list of mathematical notations appeared in this paper:

Symbols	Definition
$\mathbb{R}^+$	$[0,\infty)$
D	differential operator
$\mathbf{P}(A)$	probability of event A
$\mathbf{E}[X]$	expectation of random variable X
$\mathbf{V}[X]$	variance of X
$\mathbf{m}_n[X]$	n^{th} order moment of X
$\mathbf{c}_n[X]$	n^{th} order central moment of X
$\mathbf{B}(x,r)$	ball of radius r centered at x (in d dimension)
$\mathbf{1}_{\{A\}}(x)$	indicator function
$\mathbf{C}[X,Y]$	covariance of two random variable X,Y
$Q(u)$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-\frac{x^2}{2}}\,\mathrm{d}\,x$
$\overline{Q}(u)$	$\frac{1}{\sqrt{2\pi}}\int_{u}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}\,x$
$g(u)$	$(1+u)\ln(1+u)-u$

Table I
MATHEMATICAL NOTATIONS.

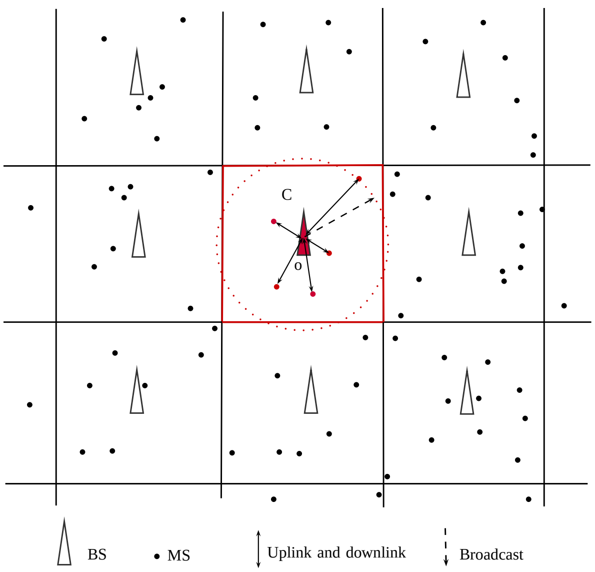

Figure 1. Power consumption model.

II. SYSTEM MODEL

A. Model for energy consumption

We suppose that there is a cellular network with multiple base stations on $\mathbb{R}^d$ and there is a base station located at the origin o administering a geographical region C around o. Practically speaking, base stations are deployed on a plane. Hence $d=2$. However for the sake of generality, we compute the formulas for any value of d. Note that considering $d=3$ can be interesting if we study the deployment of wireless access points in a building. We assume that there exists $0<R_1<R$ such that $\mathbf{B}(o,R_1)\subset C\subset \mathbf{B}(o,R)$ and C is convex and compact. We define $R_{inf}=\inf_R\{C\subset\mathbf{B}(o,R)\}$ For a given spatial configuration of active users on $\mathbb{R}^d$ at an instance, denoted by η, users located inside C will be served by o, users outside this region will be served by another base station (or in outage regime). The power consumed by the battery of the base station o can be divided into two parts:

- the power dedicated to transmit, receive, decode and encode the signal of any active user. The cumulated power over the whole configuration is then of the form $\sum_{x\in\eta}\phi(x)$, where ϕ is a function to be defined later.
- the power dedicated to broadcast messages. In order to guarantee that all active users receive these messages, the power must be such that the farthest user in the cell is within the reception range (if the system performs power control) or all the cell is within reception range (if the system does not performs power control). Thus, the power is a function of $\max_{x\in\eta\cap C}|x|$ where $|x|$ is the Euclidean norm of x (the power is equal to 0 if $\eta\cap C=\emptyset$). This function is constant if power control is not performed.

It follows that the total consumed power is given by:

$$P(\eta)=P_A(\eta)+P_B(\eta), \tag{1}$$

where $P_A(\eta)=\sum_{x\in\eta}\phi(x)$ and $P_B(\eta)=\overline{\psi}(\|\eta\|)$, $\|\eta\|=\max_{x\in\eta\cap C}|x|$ if $\eta\cap C\neq\emptyset$ and $\|\eta\|=0$ if $\eta\cap C=\emptyset$ (the subscript A stands for "additive" and B stands for "broadcast"). For a very simple propagation model (without fading and shadowing), the Shannon's formula states that for a receiver located at x, the transmission rate is given by

$$W\log_2(1+P_e l(x)),$$

where W is the bandwidth, P_e is the transmitted power and $l(x)$ is the pathloss function. Generally, the function $l:\mathbb{R}^d\to[0,\infty]$ takes the form $\overline{l}(|x|)$ where $\overline{l}:[0,\infty)\to[0,\infty]$ is a non decreasing function. This implies that in order to guarantee a minimum rate at position x, P_e must be proportional to $l(x)$. Thus, it is sensible to choose ϕ as

$$\phi(x)=a.\overline{l}(|x|)\mathbf{1}_{\{x\in C\}}$$

with $a>0$. The function ψ is chosen as

$$\psi(x)=\begin{cases} b.l(|x|)\mathbf{1}_{\{x\in C\}}, & \text{if power control is performed;}\\ b.l(R_{inf}), & \text{if power control is not performed.}\end{cases}$$

We can divide models for path loss into two categories:

- singular path loss model $l(x)=K\,|x|^{-\gamma}$ where γ is the exponent path loss parameter and K is a positive constant.
- non-singular path loss models like $l(x)=K(r_0\vee|x|)^{-\gamma}$ or $l(x)=K(1+|x|^{-\gamma})^{-1}$.

More generally, we make the following assumption:

Assumption 1. *The transmitted power depends only on the distance to the base station $\phi(x)=\overline{\phi}(|x|)\mathbf{1}_{\{x\in C\}}$. Furthermore, $\overline{\phi}$ and $\overline{\psi}$ are continuous non decreasing function on $\mathbb{R}^+$.*

We denote $\psi(x)=\overline{\psi}(|x|)\mathbf{1}_{\{x\in C\}}$. This implies that ϕ and ψ are always bounded function. Apart from the above model for power consumption, we can define a power consumption as a general functional depending on the configuration of users $P_G:\Omega^{\mathbb{R}^d}\longrightarrow[0,\infty]$.

If $\omega=(\omega_t,t\geq 0)$ is a process of time varying configurations, the total consumed energy between time 0 and time T

is given by

$$J_T := J_T(\omega, T) = \int_0^T P(\omega_s)\mathrm{d}s\cdot$$

As previously, we also define J_A and J_B by:

$$J_A := J_A(\omega, T) = \int_0^T P_A(\omega_s)\mathrm{d}s$$

and

$$J_B := J_B(\omega, T) = \int_0^T P_B(\omega_s)\mathrm{d}s\cdot$$

The same definition for $J_G(\omega, T)$ if the system applies power consumption $P_G(.)$.

Also we denote $C(r) = C \cap \mathbf{B}(o, r)$ and $\overline{C}(r) = C \cap \overline{\mathbf{B}}(o, r)$. For a configuration ν, we denote x_ν the point of ν such that $|x_\nu| = \|\nu\|$ (if there are more than one point then x_{nu} is randomly chosen among these points).

B. Model for mobility of users

In this section, we introduce the mobility models for users.

Consider the functional space $D(\mathbb{R}^+, \mathbb{R}^d)$ of rcll functions on $\mathbb{R}^d$ equipped with the Skorohod topology(see, for example [6], page 369). It is well known that $D(\mathbb{R}^+, \mathbb{R}^d)$ is a Polish space. The subset $D_0(\mathbb{R}^+, \mathbb{R}^d) = \{f \in D(\mathbb{R}^+, \mathbb{R}^d), f(0) = o\}$ equipped with the Skorohod topology is also a Polish space. We consider a probability distribution $\mathbf{P}_M$ of a random variable $M = (M(t), t \in \mathbb{R}^+)$ defined on the associated Borelian $\sigma-$field of the space $D_0(\mathbb{R}^+, \mathbb{R}^d)$. Each realization of M can be represented as a rcll trajectory of on $\mathbb{R}^d$. Also, this probability is completely determined by the distributions of finite marginal distributions $\mathbf{P}\left(M(t_1) \in ., ..., M(t_n) \in .\right)(t_1, ..., t_n > 0)$. In some situation, for convenience we can assume that $M(t) = o$ for $t < 0$. M is said to satisfy the property $\mathbf{T}$ if $\mathbf{P}(M(t_1) = M(t_2)) = 0$ for any $0 \le t_1 < t_2$.

If mobility is considered, then each user is associated with a mobility process on $\mathbb{R}^d$. We make the following assumption:

Assumption 2.

- *Motion trajectories of users are i.i.d mutually independent and have the same distribution as that of M.*
- *Motion trajectories of users do not depend on the initial position of users.*

More precisely, consider a user i initially arriving at x_i is associated with an independent version of M, namely M_i and an arrival time T_i. This user will move during its sojourn along M_i, i.e the position of this user at time $t \ge T_i$ is $x + M_i(t - T_i)$. The random process $(M_i)_{i\in\mathbf{N}}$ are mutually independent. Examples for mobility model are as follows:

- Motionless users: $M(t) = o, \forall t \in \mathbb{R}$.
- Brownian motion users: $M(t) = c(t)B_d(t)$ where $c(t) \in \mathbb{R}$ is a continuous function in $\mathbb{R}^+$ and B_d is a standard Brownian motion on $\mathbb{R}^d$.
- Completely aimless users: $M(t) = tv$ where the speed of user v is random whose direction is chosen randomly and uniformly over the d-dimensional unit sphere and $|v|$ is a positive random variable.
- Combination of two above models: $M(t) = tv + c(t)B_d(t)$.
- High mobility regime: let $\epsilon > 0$ be a small parameter, the high mobility regime consists of considering the mobility process $(M/\epsilon)(t) = M(t)/\epsilon$ and we want to study the behavior of the system when $\epsilon \to \infty$. The high mobility regime of the completely aimless mobility model with constant speed $|v|$ is the same as considering $|v| \to \infty$, i.e when the user's speed is very high.

C. ON-OFF model for users' activity

An ON-OFF process on the real line alternates between values 1 (for on state) and 0 (for OFF state). ON-periods (and OFF-periods) are i.i.d positive random variables. Furthermore, the sequences of ON-periods and OFF-periods are independent. Each realization of an ON-OFF source is a rcll function. An ON-OFF process is called exponential if ON-periods and OFF-periods are exponential distributed.

More precisely, we consider an ON-OFF sources $(I(t), t \in \mathbb{R})$ such that the ON-periods are continuous positive random variable of mean $\mu_1^{-1} > 0$ and the OFF-periods are continuous positive random variable of mean $\mu_0^{-1} > 0$ and denote by U and V the generic ON-period and OFF-period. We can write:

$$I(t) = \sum_{i=-\infty}^{\infty} \mathbf{1}_{\{T_{2i} \le t < T_{2i+1}\}}.$$

where $... < T_3 < T_2 < T_{-1} < T_0 < T_1 < T_2 < T_3 < ...$ such that $(T_{2i} - T_{2i-1})_{i=-\infty}^{\infty}$ are i.i.d and have the same distribution as V, and $(T_{2i+1} - T_{2i})_{i=-\infty}^{\infty}$ are i.i.d and have the same distribution as U. We can assume that $T_0 \vee T_1 \ge 0$. I can be seen as a random variable on $D(\mathbb{R}, \mathbb{R})$ with probability measure $\mathrm{d}\,\mathbf{P}_I$. We assume that I is stationary. From [7] for example, we have $\mathbf{P}(I(t) = 1) = \frac{\mu_0}{\mu_0+\mu_1} \triangleq \pi_1$ and $\mathbf{P}(I(t) = 0) = \frac{\mu_1}{\mu_0+\mu_1} \triangleq \pi_0$ for all t. Furthermore, we have

$$\mathbf{P}\left(I(t) = 1 \forall t \in [0, u)\right) = \frac{\pi_1}{\mathbf{E}\left[U\right]} \int_u^\infty \mathbf{P}\left(U > s\right)\,\mathrm{d}\,s.$$

for all $u > 0$.

We make the following assumptions:

Assumption 3. *The positions of users at $t = 0$ follow a Poisson point process $N = \{X_i\}_{i\ge1}$ of intensity measure $\lambda\,\mathrm{d}\,x$. User i is associated with an ON-OFF process of activity $(I_i(t), t \in \mathbb{R})$,i.e users are active during their ON-periods and are inactive during their OFF-periods. The activity processes of users are assumed to be i.i.d and have the same distribution as that of $(I(t), t \ge 0)$.*

Following the above assumptions, the configuration of active users at time t is

$$\omega_t^M = \sum_{i\ge1} \mathbf{1}_{\{I_i(t)=1\}} \delta_{X_i + M_i(t)}.$$

The system can be described by a Poisson point process on $\mathbb{R}^d \times D(\mathbb{R},\mathbb{R}) \times D(\mathbb{R}^+,\mathbb{R}^d)$

$$\Phi^{I,M} = \{(X_i, I_i, M_i)\}_{i\geq 1}.$$

of intensity $\lambda \,\mathrm{d}\, x \times \mathrm{d}\,\mathbf{P}_I \times \mathrm{d}\,\mathbf{P}_M$. The consumed energy is defined in the same way as in the previous subsection for the time varying configuration process $\omega^M = (\omega^M(t), t \geq 0)$. In particular, the additive part of consumed energy can be rewritten as:

$$J_A(\omega^M, T) = \sum_{i\geq 1} \int_0^T I_i(t)\phi(X_i + M_i(t)) \,\mathrm{d}\, t$$

and the broadcast part is:

$$J_B(\omega^M, T) = \int_0^T \overline{\psi}(\|\omega_t^M\|) \,\mathrm{d}\, t.$$

The total consumed energy is $J_T(\omega^M,T) = J_A(\omega^M,T) + J_B(\omega^M,T)$.

When users are motionless, i.e $M_i(t) = o$, the system is described as a Poisson point process

$$\Phi^I = \{(X_i, I_i)\}_{i\geq 1}.$$

of intensity measure $\lambda \,\mathrm{d}\, x \times \mathrm{d}\,\mathbf{P}_I$. In this case, we drop the superscript. Thus, the configuration of users at time t is:

$$\omega_t = \sum_{i=1}^{n} \mathbf{1}_{\{I_i(t)=0\}}\delta_{X_i}.$$

The additive part of consumed energy is

$$J_A(\omega, T) = \sum_{i\geq 1} \phi(X_i) \int_0^T I_i(t) \,\mathrm{d}\, t$$

and the broadcast part is:

$$J_B(\omega, T) = \int_0^T \overline{\psi}(\|\omega_t^M\|) \,\mathrm{d}\, t.$$

The total consumed energy in this case is $J_T(\omega,T) = J_A(\omega,T) + J_B(\omega,T)$. In the next two sections we present analytical results on the motionless case and the general case.

III. Motionless case

In this section, we assume that users are motionless. We derive analytical expressions for the moments of $J_A(\omega,T), J_B(\omega,T), J_T(\omega,T)$ in this case. Let $G(h) = \int_{\mathbb{R}^d} h(x) \,\mathrm{d}\, x$ for $h : \mathbb{R}^d \to \mathbb{R}$.

Theorem 1. *The moments of $J_A(\omega,T)$ and the central moments of $J_A(\omega,T)$ are given by:*

$$\mathbf{m}_n [J_A(\omega,T)] = B_n \left(\lambda \mathbf{m}_1 [A(T)] G(\phi), ..., \lambda \mathbf{m}_n [A(T)] G(\phi^n)\right),$$

and

$$\mathbf{c}_n [J_A(\omega,T)] = B_n \left(0, \lambda \mathbf{m}_2 [A(T)] G(\phi^2), ..., \lambda \mathbf{m}_n [A(T)] G(\phi^n)\right).$$

In particular, the expectation of $J_A(\omega,T)$ is given as:

$$\mathbf{E}[J_A(\omega,T)] = \lambda\pi_1 \int_{\mathbb{R}^d} \phi(x) \,\mathrm{d}\, x. \tag{2}$$

Proof: The theorem is derived from theorem 13, see more in details in [8] (section 7.3, theorem 31). ■

We see that, from the above theorem, the expectation of $J_A(\omega,T)$ depends on the distribution of ON-periods and OFF-periods only by the activity rate μ_0/μ_1.

Applying theorem 14, an error bound for Gaussian approximation of $J_A(\omega,T)$ is found as follows:

Theorem 2. *Let $\overline{J_A}(\omega,T) = \frac{J_A(\omega,T) - \mathbf{E}[J_A(\omega,T)]}{\mathbf{V}[J_A(\omega,T)]}$ then for any u we have:*

$$\left|\mathbf{P}\left(\overline{J_A}(\omega,T) > u\right) - \overline{Q}(u)\right| \leq \frac{\mathbf{m}_3 [A(T)] G(\phi^3)}{\sqrt{\lambda}\left(\mathbf{m}_2 [A(T)] G(\phi^2)\right)^{\frac{3}{2}}}.$$

The above bound decays as $\Theta\left(\frac{1}{\sqrt{\lambda}}\right)$ as $\lambda \to \infty$. As shown in Lemma 20, $T^2 \geq \mathbf{m}_2 [A(T)] \geq \pi_1^2 T^2$ and $\mathbf{m}_3 [A(T)] \leq T^3$, so the bound decays as $O(1)$ as $T \to \infty$. We obtain a less sharp bound but depending only on the activity rate π_1 but not on the distribution of ON-periods and OFF-periods and on T:

$$\left|\mathbf{P}\left(\overline{J_A}(\omega,T) > u\right) - \overline{Q}(u)\right| \leq \frac{\int_{\mathbb{R}^d} \phi^3(x) \,\mathrm{d}\, x}{\pi_1^3 \sqrt{\lambda}\left(\int_{\mathbb{R}^d} \phi^2(x) \,\mathrm{d}\, x\right)^{\frac{3}{2}}}.$$

As already noted in the case of exponential ON-OFF source, when T goes to infinity, $\mathbf{m}_2 [A(T)] \sim \pi_1^2 T^2$ and $\mathbf{m}_3 [A(T)] \sim \pi_1^3 T^3$. Consequently, in this case the bound has the following limit when $T \to \infty$:

$$\frac{\int_{\mathbb{R}^d} \phi^3(x) \,\mathrm{d}\, x}{\sqrt{\pi_1 \lambda}\left(\int_{\mathbb{R}^d} \phi^2(x) \,\mathrm{d}\, x\right)^{\frac{3}{2}}}.$$

Theorem 3. *The joint distribution of $(\|\omega_{t_1}\|, ..., \|\omega_{t_n}\|)$ is given by:*

$$F_{(\|\omega_{t_1}\|,...,\|\omega_{t_n}\|)}(u_1,...,u_n) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1\leq k_1 < ... < k_i \leq n} e^{-\pi_{1,1,...,1}(t_{k_1},...,t_{k_i})\Lambda(\overline{C}(\max\{u_{k_1},...,u_{k_i}\}))}.$$

In particular,

$$F_{\|\omega_t\|}(u) = e^{-\pi_1 \Lambda(\overline{C}(u))}. \tag{3}$$

The n^{th} order moment of $J_B(\omega,T)$ are given by:

$$\mathbf{m}_n [J_B(\omega,T)] = \int_{[0,T)^n} \mathrm{d}\, t_1 ... \,\mathrm{d}\, t_n \int_{(\mathbb{R}^+)^n} \overline{\psi}(u_1)...\overline{\psi}(u_n) \,\mathrm{d}\, F_{(\|\omega_{t_1}\|,...,\|\omega_{t_n}\|)}(u_1,...,u_n).$$

Proof: See [8] (section 7.3, theorem 33). ■

We have obtained closed form formulas for $\mathbf{V}[J_B(\omega,T)]$ and $\mathbf{V}[J_T(\omega,T)]$, however it requires to solve triple integrals. We now present bounds for $\mathbf{V}[J_B(\omega,T)]$ and $\mathbf{V}[J_T(\omega,T)]$, which requires only one easily computed integral.

Theorem 4. *We have:*

$$\mathbf{V}[J_B(\omega,T)] \leq \lambda \mathbf{m}_2 [A(T)] G(\psi^2), \tag{4}$$

and

$$\lambda \mathbf{m}_2 [A(T)]\, G(\phi^2) \leq \mathbf{V}[J_T(\omega,T)] \leq \lambda \mathbf{m}_2 [A(T)]\, G((\phi+\psi)^2).$$

Proof: The theorem is derived from theorem 16 and theorem 17, see [8] (section 7.3, theorem 35). ∎

The following theorem gives an upper bound on the distribution of $J_T(\omega,T)$:

Theorem 5. *Assume that $\phi(x)+\psi(x) \leq K$ for all $x \in \mathbb{R}^d$, let*

$$\alpha^2 = \mathbf{m}_2 [A(T)] \int_{\mathbb{R}^d} (\psi(x)+\phi(x))^2 \, \mathrm{d}\, x$$

then

$$\mathbf{P}(J_T(\omega,T) > \mathbf{E}[J_T(\omega,T)] + u) \leq \exp\left\{-\frac{T^2K^2}{\lambda^2\alpha^2} g\left(\frac{uTK}{\lambda^2\alpha^2}\right)\right\}$$

for all $u > 0$.

Proof: The theorem is derived from theorem 18, see also [8] (section 7.3, theorem 36). ∎

IV. Impact of mobility

We have, by the lemma 19, when users move, the spatial distribution of users remain the same as without mobility.

Following the previous lemma, we can prove that in both cases with and without mobility, the consumed energy has the same expectation, which is somewhat surprising.

Theorem 6. *For any power allocation policy P_G, and for any mobility model M, the expectation of energy consumed is the same as in motionless case, i.e:*

$$\mathbf{E}\left[J_G(\omega^M,T)\right] = \mathbf{E}[J_G(\omega,T)] \cdot$$

In particular $\mathbf{E}\left[J_A(\omega^M,T)\right] = \mathbf{E}[J_A(\omega,T)]$, $\mathbf{E}\left[J_B(\omega^M,T)\right] = \mathbf{E}[J_B(\omega,T)]$ and $\mathbf{E}\left[J_T(\omega^M,T)\right] = \mathbf{E}[J_T(\omega,T)]$.

Proof: See [8] (section 7.4, theorem 38). ∎

The distribution of consumed energy in the mobility case is clearly not the same as in the motionless case. Let

$$\digamma_n^M(f,T) = \int_{\mathbb{R}^d} \mathbf{E}\left[\left(\int_0^T f(x+M(t))I(t)\ \mathrm{d}\,t\right)^n\right] \mathrm{d}\,x.$$

In the following theorem, we characterize the impact of mobility on the distribution of the additive part of consumed energy:

Theorem 7. *The moments of $J_A(\omega^M,T)$ are given by*

$$\mathbf{m}_n\left[J_A(\omega^M,T)\right] = B_n(\lambda\digamma_\phi^M(T,1), \lambda\digamma_\phi^M(T,2), ..., \lambda\digamma_\phi^M(T,n))$$

and

$$\mathbf{c}_n\left[J_A(\omega^M,T)\right] = B_n(0, \lambda\digamma_\phi^M(T,2), ..., \lambda\digamma_\phi^M(T,n))$$

Mobility reduces moments of J_A; i.e

$$\mathbf{m}_n\left[J_A(\omega^M,T)\right] \leq \mathbf{m}_n[J_A(\omega,T)]$$

and

$$\mathbf{c}_n\left[J_A(\omega^M,T)\right] \leq \mathbf{c}_n[J_A(\omega,T)].$$

Furthermore, $J_A(\omega^M,T)$ is Laplace-smaller than $J_A(\omega,T)$, i.e:

$$\mathbf{E}\left[\exp\left\{\alpha J_A(\omega^M,T)\right\}\right] \leq \mathbf{E}[\exp\{\alpha J_A(\omega,T)\}] \tag{5}$$

for all $\alpha \in \mathbb{R}$. If M has the property $\mathbf{T}$ then the central moments of J_A goes to 0 in high mobility regime; i.e

$$\begin{aligned}
\mathbf{c}_n\left[J_A(\omega^{M/\epsilon},T)\right] &\to 0, \\
\mathbf{m}_n\left[J_A(\omega^{M/\epsilon},T)\right] &\to \left(\pi_1\lambda\int_{\mathbb{R}^d}\phi(x)\ \mathrm{d}\,x\right)^n \\
&= (\mathbf{E}[J_A(\omega,T)])^n
\end{aligned}$$

as $\epsilon \to 0\cdot$.

Proof: The expressions for the moments and central moments come from theorem 13. The Laplace-smaller property is proved by the fact that $e^{\alpha u}$ is a convex function for all $\alpha \in \mathbb{R}$. The moments reducing property and the convergence to 0 is derived thanks to lemma 22. For more details, see [8] (section 7.3, theorem 41). ∎

The following theorem gives impact of high mobility on the distribution of $J_B(\omega^M,T)$ and $J_T(\omega^M,T)$:

Theorem 8. *The variance of $J_B(\omega^M,T)$ and $J_T(\omega^M,T)$ are bounded as follows:*

$$\begin{aligned}
\mathbf{V}\left[J_B(\omega^M,T)\right] &\leq \lambda\digamma_\psi^M(T,2) \\
\lambda\digamma_\phi^M(T,2) \leq \mathbf{V}\left[J_T(\omega^M,T)\right] &\leq \lambda\digamma_{\phi+\psi}^M(T,2)\cdot
\end{aligned}$$

*Therefore, if M has the property **T** then in high mobility regime, the variance of J_B and J_T tends to 0, i.e*

$$\mathbf{V}\left[J_B(\omega^{M/\epsilon},T)\right], \mathbf{V}\left[J_T(\omega^{M/\epsilon},T)\right] \to 0 \text{ as } \epsilon \to 0\cdot$$

Proof: The theorem is proved thanks to theorem 17 and lemma 22. See also [8] (section 7.4, theorem 42). ∎

The above results say that, when users move the total consumed energy by a base station does not change *in average*, and the moments and central moments of the additive part are reduced. Moreover, when users move very fast, the consumed energy during a time period is almost constant. We can see this fact as a consequence of weak central limit theorem. When users move faster, the configuration of users takes more "value" on $\Omega^{\mathbb{R}^d}$ during a same period of time, thus converge faster to the mean.

We find an error bound for Gaussian approximation of $J_A(\omega^M,T)$ as follows:

Theorem 9. *Let $\overline{J_A}(\omega^M,T) = \frac{J_A(\omega^M,T)-\mathbf{E}\left[J_A(\omega^M,T)\right]}{\mathbf{V}[J_A(\omega^M,T)]}$ then:*

$$\left|\mathbf{P}\left(\overline{J_A}(\omega^M,T) > u\right) - \overline{Q}(u)\right| \leq \frac{\digamma_\phi^M(T,2)}{\sqrt{\lambda}\left(\digamma_\phi^M(T,3)\right)^{3/2}}$$

Proof: This is a consequence of theorem 14. See also [8] (section 7.4, theorem 43). ∎

Theorem 10. *Assume that $\phi(x)+\psi(x)\le K$ for all $x\in\mathbb{R}^d$, then*

$$\mathbf{P}\left(J_T(\omega^M,T)>\mathbf{E}\left[J_T(\omega^M,T)\right]+u\right)\le \exp\left\{-\frac{T^2K^2}{\lambda F^M_{\phi+\psi}(T,2)}g\left(\frac{uTK}{F^M_{\phi+\psi}(T,2)}\right)\right\}$$

for all $u>0$.

Proof: This is a consequence of theorem 18. See also [8] (section 7.4, theorem 43). ∎

V. Completely aimless mobility model

In this section, we consider the completely aimless mobility model M with constant speed, i.e $M(t)=tv$ where $|v|=constant$ and the direction of v is uniformly distributed. The following theorem characterizes explicitly the convergent rate to 0 of central moments of J_A, as well as the variance of J_B and J_T in function of user speed $|v|$.

Theorem 11. *Consider the completely aimless mobility model $M(t)=vt$ with constant speed $|v|$ then*

1) *$\mathbf{V}\left[J_A(\omega^M,T)\right]$ and $\mathbf{V}\left[J_T(\omega^M,T)\right]$ decay as $\Theta(\frac{1}{|v|})$ as $|v|\to\infty$ and decay as $O(T)$ as $T\to\infty$. The coefficient of variations $CV(J_A(\omega^M,T))$ and $CV(J_T(\omega^M,T))$ decay as $\Theta(\frac{1}{\sqrt{|v|}})$ as $|v|\to\infty$ and $\Theta(\frac{1}{\sqrt{T}})$ as $T\to\infty$.*
2) *The error bound of Gaussian approximation in Theorem 9 decays as $\Theta(|v|^2)$ as $|v|\to\infty$ and decays as $\Theta(\frac{1}{\sqrt{T}})$ as $T\to\infty$.*

Proof: From the assumption 1 we have $\phi(x)\le c_1$ for all $x\in C$, $\phi(x)\ge c_2$ for all $x\in C/B(o,\frac{R_1}{4})$ and $0\le\psi(x)\le c_3$ for all $x\in C$ for some finite positive constants c_1,c_2,c_3. Now, according to Theorem 8:

$$\begin{aligned}\lambda F^M_{\phi}(T,n)&=\mathbf{V}\left[J_A(\omega^M,T)\right]\\&\le\mathbf{V}\left[J_T(\omega^M,T)\right]\quad\le\quad\lambda F^M_{\phi+\psi}(T,n).\end{aligned}$$

The results then follows from Lemma 23. ∎

We also see that, if $|v|$ is small, the variance of $J_T(\omega^M,T)$ is proportional to $\frac{T}{|v|}$ while in the motionless case, it is $\Theta(T^2)$. We also notice that mobility makes the Gaussian approximation of J_A more accurate when T is large as the bound decays as $\Theta(\frac{1}{\sqrt{T}})$ instead of $\Theta(1)$ in the motionless case. In the motionless case the position of users are always fixed over time, only their state change, thus the configuration of active users can take only some possible values on $\Omega^{\mathbb{R}^d}$. So it is intuitive that one cannot guarantee that the Gaussian approximation is good if T is large. On the contrary, in the mobility case, the configuration of active users can take all possible values on $\Omega^{\mathbb{R}^d}$. When T grows larger, it take more values. Thus, Gaussian approximation is better when T grows larger. Quite surprisingly, when $|v|$ is large, the variance of J_A tends to 0 but the bound on Gaussian approximation does not decrease.

Remark 1. *From the above proof and using the properties of Bell polynomial, we can prove that $\mathbf{c}_3\left[J_A(\omega^M,T)\right]=\Theta(\frac{1}{|v|^2})$, $\mathbf{c}_4\left[J_A(\omega^M,T)\right]=\Theta(\frac{1}{|v|^2})$, $\mathbf{c}_5\left[J_A(\omega^M,T)\right]=\Theta(\frac{1}{|v|^3})$,...*

VI. Applications

A. Dimensioning optimal cell size

Consider an operator aiming to design the optimal cell radius R to cover a region of total area (volume) $S\in\mathbb{R}^d$. We assume that the cells are circular. The average total cost of the network is assumed to be the sum of the operation cost during the life time of the network (says T) and the cost of facilities (base stations). We assume a fixed cost of base station regardless of its transmission range. The number of base station is then proportional to $\frac{S}{R^d}$, say $\frac{C}{R^d}$, so the installation cost of base stations is $\frac{c_1}{R^d}$ with $c_1>0$. The operation cost is assumed to be proportional to the consumed energy.

We assume that $l(x)=K.|x|^\gamma$, $\phi(x)=a.l(x)\mathbf{1}_{\{x\in C\}}$, $\psi(x)=b.l(R)$ with $K>0$ and furthermore $\gamma\ge d$. From results presented in previous sections and from lemma 24, the expectations of the two parties of energy consumed by a singe base station are $\mathbf{E}\left[J_A(\omega,T)\right]=\lambda Ta'R^{\gamma+d}$ and $\mathbf{E}\left[J_B(\omega,T)\right]=Tb'R^\gamma$ with $a',b'\ge 0$. The energy consumed by the network during its operating time is:

$$\frac{C}{R^d}\left(a'\lambda TR^{d+\gamma}+b'TR^\gamma\right)=a_1\lambda TR^\gamma+b_1TR^{\gamma-d}.$$

This is a increasing function of R, which means that small cell systems will consume less energy than larger cell system. The average total cost for the network is then

$$Cost(R)\quad=\quad a_1\lambda TR^\gamma+b_1TR^{\gamma-d}+\frac{c_1}{R^d}.$$

It is interesting for operator to find the optimal R in order to minimize the cost function. As $Cost(R)>0$ for all $R>0$ and $\lim_{R\to0}Cost(R)=\lim_{R\to\infty}Cost(R)=\infty$ there exists a minimum for $Cost$. By differentiation, the optimal cell radius is the unique positive solution of the following equation:

$$a_1\lambda\gamma TR_{opt}^{\gamma+d}+b_1T(\gamma-d)R_{opt}^\gamma=dc_1$$

As the RHS is increasing in T, the optimal cell radius R_{opt} must be a decreasing function of T. This reveals a characteristic of the optimal choice of cell radius. In the economical point of view, to operate a network with longer life time it is preferable to exploit smaller cells system.

If $b_1=0$, i.e the broadcast part of transmitted power is small comparing to the additive part, then the problem reduces to minimizing $a_1\lambda TR^\gamma+\frac{c_1}{R^d}$. Simple manipulations yields:

$$R_{opt}=\left(\frac{dc_1}{\gamma\lambda a_1T}\right)^{\frac{1}{\gamma+d}}.$$

That is to say theoretically the optimal cell radius is proportional to $(\lambda T)^{-\frac{1}{\gamma+d}}$.

B. Dimensioning cell battery

The proposed model can be used to dimension sites that do not have access to power supply facilities. In this situation, operators have to replace or reload base station's battery for each period T. We want to determine the energy level α of battery so that the probability of running out of energy before replacement(or reloading) be smaller than some threefold $\epsilon \ll 1$. We use results derived in the previous sections to find α.

Assume the mobility model M and that we are able to estimate $F_{\phi}^{M}(T,2)$ and $F_{\phi+\psi}^{M}(T,n)$. The problem is to find α such that:

$$\mathbf{P}\left(J_A(\omega^M,T) > \alpha\right) < \epsilon.$$

From theorem 10, we deduce that the following value is a solution:

$$\alpha_1 = T\pi_1 + \frac{u_1 F_{\phi+\psi}^{M}(T,2)}{TK}$$

where $u_1 > 0$ is the unique solution of the following equation:

$$g(u_1) = -\frac{\lambda F_{\phi+\psi}^{M}(T,2)\ln\epsilon}{T^2K^2}.$$

Now applying theorem 9, we can find another solution. Assume that $\psi(x) \le K_B$ for all $x \in C$ and

$$\frac{F_{\phi}^{M}(T,2)}{\sqrt{\lambda}\left(F_{\phi}^{M}(T,3)\right)^{3/2}} < \epsilon$$

then $J_T(\omega^M,T) \le J_A(\omega^M,T) + K_BT$ and:

$$\mathbf{P}\left(J_T(\omega^M,T) > \alpha\right) \le \mathbf{P}\left(J_A(\omega^M,T) > \alpha - K_BT\right).$$

Therefore, the following value is another solution:

$$\alpha_2 = \lambda G(\phi) + u_2\sqrt{\lambda F_{\phi}^{M}(T,2)} + K_BT$$

where u_2 is the unique positive number such that:

$$\overline{Q}(u_2) = \epsilon - \frac{F_{\phi}^{M}(T,2)}{\sqrt{\lambda}\left(F_{\phi}^{M}(T,3)\right)^{3/2}}.$$

VII. Conclusion

Throughout this paper, we have presented a new analysis of the energy consumed by a single base station in a cellular network. we have assumed that each user is associated with an ON-OFF process of activity. We have derived analytical expressions for the distribution of energy consumed by a base station. We have found that, with or without mobility, the base station is expected to consume the same amount of energy in average. We have proved that mobility reduced moments of the additive part of energy. We have also proved that high mobility leads the variance of energy to 0. These results are strong since they hold for any mobility model. In the case of completely aimless mobility model, we have characterized the convergence rate to 0 of the variance, which is $\frac{1}{|v|}$. The mathematical framework presented in this paper is new and can be served to further studies.

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Appendix

Poisson point process

Let E be a $\sigma-$compact metric space (i.e E can be partitioned into a countable union of compact metric spaces) with a diffuse Radon measure ν. The space of configurations of E is the set of locally finite simple point measures

$$\Omega^E := \left\{ w = \sum \epsilon_{z_i} (\text{at most countable}), z_i \in E \right\},$$

where ϵ_z denotes the Dirac measure at $z \in E$, i.e

$$\delta_z(A) = \mathbf{1}_{\{z \in A\}}, A \in \mathcal{B}(E)\cdot$$

Here, simple measure means that $w(\{z\}) \le 1$ and locally finite measure means that $w(K) < \infty$ for all compact $K \subset E$. The configuration space Ω^E is endowed with the vague topology and its associated σ-algebra denoted by $\mathcal{F}^E$.

For convenience, it is quite often to identify an element $w = \sum \epsilon_{z_i}$ with its corresponding support, i.e the unordered set $\{z_1, ..., z_n\}, n \in N \cup \{+\infty\}$. Also, $w(A)$ counts the number of points in $A \in \mathcal{B}(E)$. The distribution of point process w is characterized by the family of finite dimensional distributions $(w(A_1), ..., w(A_n))$ where $A_1, ..., A_n$ are mutually disjoint compact subsets of E.

Definition 1. *w is a Poisson point process (PPP) of intensity ν if for all set $(A_1, ..., A_n)$ of mutually disjoint compact subsets of E:*

$$\mathbf{P}(w(A_1) = k_1, ..., w(A_n) = k_n) = \prod_{i=1}^{n} \left(e^{-\nu(A_i)} \frac{(\nu(A_i))^{k_i}}{k_i!} \right) \cdot$$

If $E = \mathbb{R}^d$; $\mathcal{B}$ is Borel algebra of $\mathbb{R}^d$ and $\nu(\mathrm{d}\, z) = \lambda \,\mathrm{d}\, z$, we will call w the homogenous Poisson point process with intensity parameter λ on $\mathbb{R}^d$.

Roughly speaking, the number of points of w falling into a subset A follows Poisson distribution of parameter $\nu(A)$, and the number of points falling into 2 disjoint subsets are independent.

LINEAR FUNCTIONAL

We call F a linear functional of w if there exists $f : E \to \mathbb{R}$ such that

$$F = \int_E f(z) \,\mathrm{d}\, w(z) = \sum_{z \in w} f(z) \cdot$$

We assume that $f \in L^1(\nu)$. In this section we are interested in the distribution of F.

Let $\mathcal{L}_w(.)$ be the Laplace functional of w, i.e:

$$\mathcal{L}_w(u) = \mathbf{E}\left[e^{-\int_E u(z) \,\mathrm{d}\, w(z)} \right] \tag{6}$$

Theorem 12. *([9]) The Laplace functional of w satisfies:*

$$\mathcal{L}_w(u) = e^{-\int_E (1 - e^{-u(z)}) \,\mathrm{d}\, \nu(z)}, \qquad u \in L^1(\nu) \cdot \tag{7}$$

From the above theorem the moment generating function (MGF) of F is expressed as follows:

$$\mathbf{E}\left[e^{\int_E f \,\mathrm{d}\, w} \right] = e^{\int_E (e^{f(z)} - 1) \,\mathrm{d}\, \nu(z)}. \tag{8}$$

The complete Bell polynomials $B_n(a_1, ..., a_n)$ are defined as follows:

$$\exp\left\{ \sum_{n=1}^{\infty} \frac{a_n}{n!} \theta^n \right\} = \sum_{n=1}^{\infty} \frac{B_n(a_1, a_2, ..., a_n)}{n!} \theta^n$$

for all $a_1, ..., a_n$ and θ such that all above terms are correctly defined.

The first four Bell complete polynomials are given as:

$$\begin{aligned} B_1(a_1) &= a_1 \\ B_2(a_1, a_2) &= a_1^2 + a_2 \\ B_3(a_1, a_2, a_3) &= a_1^3 + 3a_1 a_2 + a_3 \\ B_4(a_1, a_2, a_3, a_4) &= a_1^4 + 4a_1^2 a_2 + 4a_1 a_3 + 3a_2^2 + a_4 \end{aligned}$$

The Bell polynomials helps us to express central moments and moments of F in a simple fashion:

Theorem 13 (Generalization of Campbell's formulas)**.** *Assume that $f \in \cap_{i=1}^{n} L^i(E, \nu)$. The cumulants of $F = \int_E f \,\mathrm{d}\, w$ is $\kappa_i^F = \int_E f^i(z) \,\mathrm{d}\, \nu(z)$ $(i = 1..n)$. The moments and central moments of F are given as:*

$$\mathbf{m}_i\left[F\right] = B_i \left(\int_E f(z) \,\mathrm{d}\, \nu(z), \int_E f^2(z) \,\mathrm{d}\, \nu(z), ..., \int_E f^i(z) \,\mathrm{d}\, \nu(z) \right) \tag{9}$$

and

$$\mathbf{c}_i\left[F\right] = B_i \left(0, \int_E f^2(z) \,\mathrm{d}\, \nu(z), ..., \int_E f^i(z) \,\mathrm{d}\, \nu(z) \right) \tag{10}$$

for $i = 1, 2, ..., n$.

Proof: [8]. ∎

As a direct consequence, one can easily obtain from the above theorem two useful formulas (Campbell):

Corollary 1. *Let $F = \int_E f \,\mathrm{d}\, w$ then*

$$\mathbf{E}\left[F\right] = \int_E f(z) \,\mathrm{d}\, \nu(z), \quad f \in L^1(\nu)$$

$$\mathbf{V}\left[F\right] = \mathbf{E}\left[(F - \mathbf{E}\left[F\right])^2 \right] = \int_E f^2(z) \,\mathrm{d}\, \nu(z), \quad f \in L^2(\nu) \cdot$$

We have now expressions for moments and central moments of F. We note that the central moments of F is always non negative as f is supposed to be non negative. One can ask if there is mean to compute the tail distribution of F. Gaussian approximation seems to be a first answer one can think of due to the central limit theorem (CLT). An error bound of Gaussian approximation for sum of n i.i.d random variables is known as BerryEsseen theorem. It is possible to find an alternative version of this error bound for linear functional of PPP. Let $Q(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2} \,\mathrm{d}\, u$ be the CDF of a standard Gaussian random variable and $\overline{Q}$ be its CCDF.

Theorem 14. *([10], [8]) Consider $F = \int_E f \,\mathrm{d}\, w$ with $f \in L^2(\nu)$. Let $\overline{F} = \frac{F - \mathbf{E}[F]}{\mathbf{V}[F]}$, then:*

$$\left| \mathbf{P}(\overline{F} \le a) - Q(a) \right| \le \frac{\int_E |f(z)|^3 \,\mathrm{d}\, \nu(z)}{(\int_E f^2(z) \,\mathrm{d}\, \nu(z))^{3/2}} \cdot \tag{11}$$

We have presented the Gaussian approximation and Edgeworth expansions for linear functionals. We are now interested in upper bounds on the distribution of F, which can be called concentration inequality.

Theorem 15. *([11], [8]) Let $M, a > 0$.*

Assume that $|f(z)| \le M$ $\nu-$a.s and $f \in L^2(E, \nu)$, then

$$\mathbf{P}(F > \mathbf{E}\left[F\right] + a) \le \exp\left\{ -\frac{\mathbf{V}\left[F\right]}{M^2} g\left(\frac{a.M}{\mathbf{V}\left[F\right]} \right) \right\} \tag{12}$$

GENERAL FUNCTIONAL

We now consider a general class of functional $L^2(\Omega^E, \mathbf{P})$ (not necessarily linear). Define the difference operator D as follows:

$$D_z F(w) = F(w + \epsilon_z) - F(w), \;\; \mathrm{d}\,\mathbf{P} \times \mathrm{d}\, \nu \;\; a.e. \cdot$$

for $F \in L^2(\Omega^E, \mathbf{P})$, $w \in \Omega^E$ and $z \in E$. Note that if $F = \int_E f(z)\,\mathrm{d}\,z$ then $D_z F(w) = f(z)$.

Using the difference operator D, we can bound the variance of a functional F as follows:

Theorem 16. *([9], [11]) $\forall F \in L^2(\Omega^E, \mathbf{P})$ we have:*

$$\mathbf{V}[F] \leq \mathbf{E}\left[\int_E |D_z F|^2 \,\mathrm{d}\,z\right]\cdot \tag{13}$$

Equality occurs if and only if F is linear.

Corollary 2. *$\forall F \in L^2(\Omega^E, \mathbf{P})$ such that $|D_z F| \leq f(z)$ for some non negative measurable function $f : E \to \mathbb{R}$ for all z, w then:*

$$\mathbf{V}[F] \leq \int_E f^2(z)\,\mathrm{d}\,\nu(z)\cdot \tag{14}$$

Theorem 17. *([9],[12]) Consider two functionals $F_1, F_2 \in L^2(\Omega^E, \mathbf{P})$ (not necessarily linear). Assume that $D_z F_1, D_z F_2 \geq 0$, $\mathbf{P} \times \nu$ a.s. then*

$$\mathbf{C}[F_1, F_2] \geq 0.$$

As a consequence, $\mathbf{V}[F_1 + F_2] \geq \mathbf{V}[F_1] + \mathbf{V}[F_2]$.

Recent results concerning Edgeworth's expansion for functionals depending on Poisson point processes can be found in [13] and [8].

Theorem 18. *[11], Let $F \in domD$. Assume that $|DF| \leq M$, $\mathbf{P} \times \nu$ a.s., for some $M \geq 0$ and $\int_E |D_z F|^2 \,\mathrm{d}\,\nu(z) \leq \alpha^2, \mathbf{P}-$a.s. Then for all $u > 0$ we have*

$$\mathbf{P}(F - \mathbf{E}[F] \geq u) \leq \exp\left\{-\frac{\alpha^2}{M^2} g\left(\frac{u.M}{\alpha^2}\right)\right\} \tag{15}$$

for $M > 0$ and

$$\mathbf{P}(F - \mathbf{E}[F] \geq a) \leq \exp\left\{-\frac{\alpha^2}{2u^2}\right\} \tag{16}$$

for $M = 0$ where $g(t) = (1+t)\ln(1+t) - t$.

Lemmas

Lemma 19. *ω_t^M is a Poisson point process of intensity measure $\pi_1 \lambda \,\mathrm{d}\,x$ for all t.*

Proof: Consider the point process $\sum_{i \geq 1} \delta_{X_i + M_i(t)}$. By the displacement theorem ([14]), it is a Poisson point process of intensity measure $\mathrm{d}\,\Lambda_t(x)$ characterized by:

$$\begin{aligned}
\Lambda_t(A) &= \lambda \int_{\mathbb{R}^d} \mathbf{P}(x + M(t) \in A)\,\mathrm{d}\,x \\
&= \lambda \int_{\mathbb{R}^d} \mathrm{d}\,x \int_{\mathbb{R}^d} p_{M(t)}(y) \mathbf{1}_{\{x+y \in A\}}\,\mathrm{d}\,y \\
&= \lambda \int_{\mathbb{R}^d} p_{M(t)}(y)\,\mathrm{d}\,y \int_{\mathbb{R}^d} \mathbf{1}_{\{x+y \in A\}}\,\mathrm{d}\,x \\
&= \lambda l_d(A).
\end{aligned}$$

Thus, it is a Poisson point process of intensity $\lambda \,\mathrm{d}\,x$. Now by thinning property, $\omega_t^M = \sum_{i \geq 1} \mathbf{1}_{\{I_i(t)=1\}} \delta_{X_i + M_i(t)}$ is a Poisson point process of intensity $\pi_1 \lambda \,\mathrm{d}\,x$. ∎

Lemma 20. *We have, for all T*

$$\pi_1^n T^n \leq \mathbf{m}_n[A(T)] \leq T^n.$$

Proof: Since $A(T) \leq T$ a.s. we have $\mathbf{m}_n[A(T)] \leq T^n$. Now $\mathbf{m}_n[A(T)] \geq (\mathbf{E}[A(T)])^n = \pi_1^n T^n$ by Cauchy$-$Schwarz inequality. ∎

For a non negative function $f \in L^n(\mathbb{R}^d)$ and $t_1, ..., t_n \in \mathbb{R}$ and $f(x) = 0$ if $x \in \mathbb{R}^d / C$. Define

$$\Phi_n^M(f, t_1, ..., t_n) = \int_{\mathbb{R}^d} \mathbf{E}[f(x + M(t_1))...f(x + M(t_n))]\,\mathrm{d}\,x.$$

Lemma 21. *We have $\Phi_n^M(f, t) = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}\,x$ and:*

$$\Phi_n^M(f, t_1, ..., t_n) \leq \int_{\mathbb{R}^d} f^n(x)\,\mathrm{d}\,x\cdot$$

Moreover, if M has the property $\mathbf{T}$ and $n \geq 2$ then $\phi_n^{M/\epsilon}(f, t_1, ..., t_n) \to 0$ as $\epsilon \to 0$ with $n \geq 2$.

Proof: See [8] (section 7.4, lemma 39). ∎

Lemma 22. *We have:*

$$\digamma_n^M(f, T) \leq \mathbf{m}_n[A(T)] \int_{\mathbb{R}^d} f^n(x)\,\mathrm{d}\,x\cdot$$

If M has the property $\mathbf{T}$ then $\digamma_f^{M/\epsilon}(T, n) \to 0$ as $\epsilon \to 0$ for $n \geq 2$.

Proof: See [8] (section 7.4, lemma 40). ∎

Lemma 23. *Let $f(x)$ be a positive measurable function on $\mathbb{R}^d$ such that $f(x) = 0$ for $x \in R^d / C$, $f(x) \leq c_1$ for all $x \in C$ and $f(x) \geq c_2$ for all $x \in C / B(o, \frac{R_1}{4})$ where $c_1, c_2 > 0$ are constant then*

$$\digamma_n^M(f, T) = \Theta\left(\frac{1}{|v|^{n-1}}\right), |v| \to \infty$$

and $\digamma_n^M(f, T) = \Theta(T), T \to \infty$ for $n \geq 2$.

Proof: See [8] (section 7.4, lemma 45). ∎

Lemma 24. *Denote $V_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$ the volume of a ball of radius 1 in $\mathbb{R}^d$, and $V_d' = dV_d$ then*

$$\int_{\mathbf{B}(o,R)} |x|^k \,\mathrm{d}\,x = V_d' \frac{R^{k+d}}{k+d}$$

for all real $k > -d$.

Proof: We have:

$$\begin{aligned}
\int_{\mathbf{B}(o,R)} |x|^k \,\mathrm{d}\,x &= \int_0^R r^k \,\mathrm{d}\left(V_d r^k\right) \\
&= \int_0^R V_d' r^{k+d-1}\,\mathrm{d}\,r \\
&= V_d' \frac{R^{k+d}}{k+d}.
\end{aligned}$$

∎